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The Brink Depth of a Supercritical Overfall

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THE BRINK DEPTH OF A SUPERCRITICAL OVERFALL

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Abstract

This report contains a first order analysis of the supercritical flow in an overfall based on the use of the stream line function as the basic dependent variable. It presents approximate equations for the nappes; and the formula

$$f^*(1;\varepsilon) = \frac{1}{2}\left[\frac{1}{\sqrt{1+2\varepsilon}} + \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}}\right]$$

where

$$\varepsilon = \frac{ga}{u^2},$$

g is the gravitational acceleration, a is the upstream depth and u is the velocity there. This formula is a good approximation to the brink depth ratio b/a for the range $0 \leq \varepsilon \leq 1$.

1. Introduction

This report is concerned with the supercritical flow in a two-dimensional overfall. The analysis is based on the use of the stream line function

$$y = \overline{f}(x,\gamma) \ .$$

This function is such that if the stream function is $\psi(x,y)$ so that the stream lines are given implicitly by

$$\psi(x,y) = \gamma \ ,$$

then

$$\psi(x,\overline{f}(x,\gamma)) \equiv \gamma$$

is identically satisfied.

If the acceleration due to gravity is g, if the upstream depth is a, and if the velocity there is u, then the flow is said to be critical if $\varepsilon = ga/u^2$ is equal to 1. We suppose that ε is small so that the upstream velocity ($u >> \sqrt{ga}$) of the flow is supercritical. Such a flow can prevail in a region of space where g is small compared with a and u. It can obviously also prevail if u is sufficiently large, or if a is so small that the overfall is nearly a falling sheet.

We show below that the stream line function must satisfy a nonlinear partial differential equation defined in an infinite strip, and that it must satisfy nonlinear boundary conditions along the sides of the strip. Since these boundary conditions involve the

small parameter ε, we assume that $\overline{f}(x,\gamma)$ can be approximated, at least asymptotically, by an expansion in integral powers of ε.

The development which follows is confined to an analysis and discussion of the first order approximation and what it yields with respect to the shapes of the free surfaces and the brink depth of the overfall. A formula which is derived for the brink depth ratio, namely,

$$f^*(1;\varepsilon) = \frac{1}{2}\left[\frac{1}{\sqrt{1+2\varepsilon}} + \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}}\right]$$

appears to be a good approximation in the range $0 \leq \varepsilon \leq 1$. According to this formula the brink depth ratio for critical flow is .719. This is close to .708, the average of six approximations based on different procedures developed by various investigators.

2. Formulation

A two-dimensional waterfall or free overfall is defined by its flow in a plane perpendicular to its linear edge. Figure 2.1

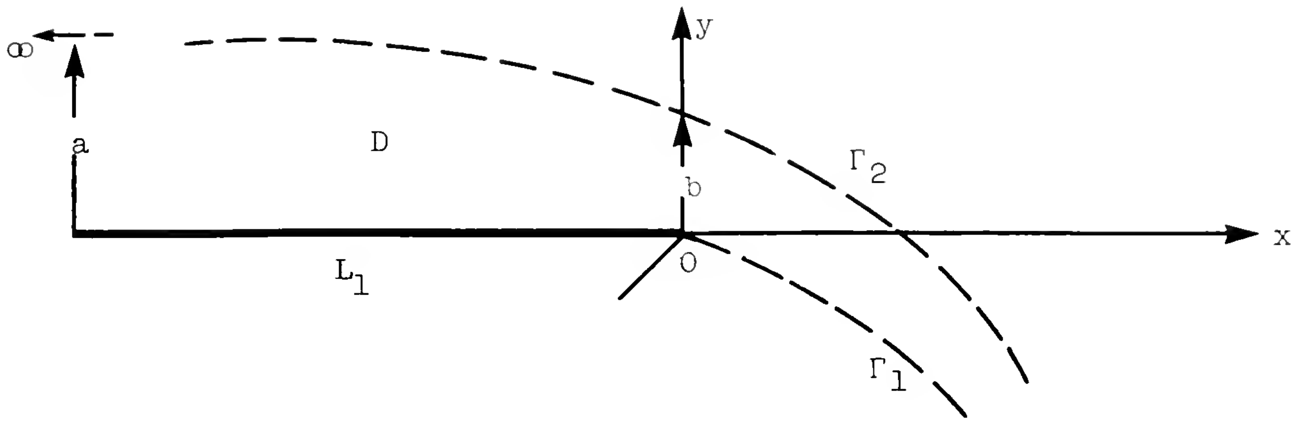

Fig. 2.1

shows such a flow referred to a rectangular coordinate system whose origin is at the brink. The negative part of the x-axis coincides with the sill or bottom L_1 and the y-axis is positive upwards. The domain D of the cross sectional flow is bounded by L_1; and the curves Γ_1 and Γ_2 which define the free surfaces or nappes of the overfall.

The difficult basic problem is to deduce the shapes of Γ_1 and Γ_2 as consequences of the assumptions that gravitational attraction is the only body force acting and that the flow is a steady irrotational one of an inviscid, incompressible fluid whose density δ is

constant. If $\underline{v}$ is the velocity vector, conservation of mass requires $\nabla \cdot \underline{v} = 0$ and this with the vorticity condition, $\nabla \times \underline{v} = 0$, insures the existence of a function of the complex variable $z = x + iy$, namely the complex velocity potential,

$$w = \phi(x,y) + i\psi(x,y) = F(z) \tag{2.1}$$

which is analytic for z in D. This function is such that the gradient of its real part gives the velocity vector $\underline{v} = \nabla \text{ Re } F(z) = \nabla\phi$; and such that its imaginary part, $\psi(x,y) = \text{Im } F(z)$, is the stream function for the flow. In other words, the horizontal and vertical velocity components are given respectively by the real and imaginary parts of

$$\overline{F'(z)} = \phi_x - i\psi_x = \psi_y + i\phi_y \ .$$

If $\underline{n}$ is the unit outward normal to the boundary of D and if s denotes arc length along the boundary, then the kinematic boundary condition requires that the normal velocity

$$\phi_n = \psi_s$$

must be zero along L_1, Γ_1 and Γ_2. It follows from this, by integration with respect to s, that L_1 plus Γ_1 constitutes a stream line which we can assume is defined by the level curve

$$\psi(x,y) = 0 \ .$$

With this, the upper nappe curve Γ_2 is defined by level curve

$$\psi(x,y) = h = au$$

where a is the upstream depth of the flow at $x = -\infty$, and u is the velocity there. In addition to the kinematic boundary condition, the energy integral of the momentum equation

$$\delta \frac{d\underline{v}}{dt} = \delta\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = \delta\underline{v}\cdot\nabla\underline{v}$$

$$= \frac{\delta}{2}\,[\nabla\underline{v}\cdot\underline{v} - \underline{v}\times\nabla\times\underline{v}]$$

$$= \frac{\delta}{2}\,\nabla\underline{v}\cdot\underline{v} = -\nabla p - g\delta\nabla y$$

namely,

$$\phi_x^2 + \phi_y^2 + \frac{2p}{\delta} + 2gy = c \tag{2.2}$$

must be satisfied along Γ_1 and Γ_2. In this equation g denotes the gravitational acceleration and p denotes the pressure in the fluid. Since the flow is supposed to be steady the constant c remains the same no matter where the left-hand side is evaluated. Therefore with

$$\phi_x(-\infty,y) = u$$

$$\phi_y(-\infty,y) = 0$$

$$p(-\infty,a) = p_a$$

we have

$$c = u^2 + 2ga + \frac{2p_a}{\delta}\,. \tag{2.3}$$

In what follows we assume that the pressure on each nappe is constant and equal to p_a.

Although the domain D is not known at the outset we can work with a fixed domain in the $w = \phi + i\psi$ plane if we introduce the inverse function

$$(2.4) \qquad z = x(\phi,\psi) + iy(\phi,\psi) = G(w) .$$

Under the mapping from the z-plane to the w-plane, D is mapped into the strip defined by

$$0 < \psi = \operatorname{Im} w < h = au ; \quad -\infty < \phi = \operatorname{Re} w < \infty ,$$

and we can choose the origin of the w-plane to be the image of the origin of the z-plane. With the origin fixed in this way the images of L_1, Γ_1 and Γ_2 are respectively M_1, Σ_1 and Σ_2 as shown in Figure 2.2. Since there are no stagnation points in the overfall, the

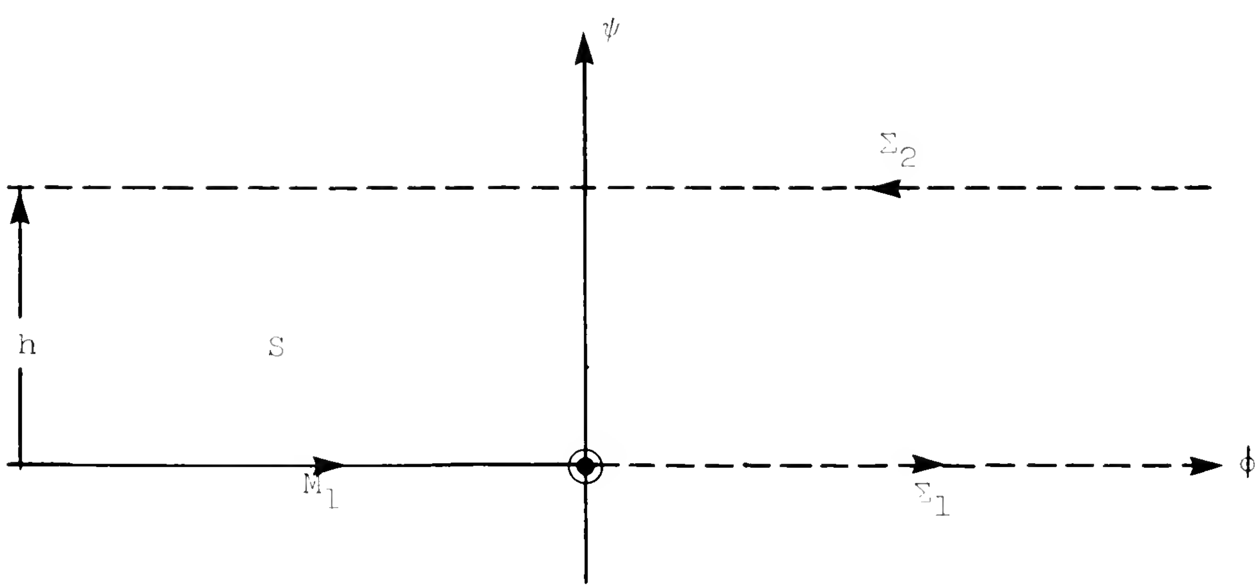

Fig. 2.2

function G(w) must be analytic in S and we have

$$\frac{dz}{dw} = G'(w) = x_\phi + iy_\phi = -ix_\psi + y_\psi$$

$$= \frac{1}{\frac{dw}{dz}} = \frac{1}{\phi_x + i\psi_x} = \frac{1}{\phi_x - i\phi_y} = \frac{\phi_x + i\phi_y}{\phi_x^2 + \phi_y^2}$$

from which

$$G'(w)\overline{G'(w)} = |G'(w)|^2 = x_\phi^2 + y_\phi^2 = \frac{1}{\phi_x^2 + \phi_y^2}$$

and

$$\phi_x = \frac{\text{Re } G'(w)}{|G'(w)|^2} ; \quad \phi_y = \frac{\text{Im } G'(w)}{|G'(w)|^2} .$$

These relations with (2.2) and (2.3) show that G(w) must satisfy the following boundary conditions:

$$\text{Im } G'(\phi) = 0 , \qquad \phi < 0 ;$$

$$\frac{1}{|G'(\phi)|^2} + 2g \text{ Im } G(\phi) = u^2 + 2ga , \qquad \phi > 0 ;$$

$$\frac{1}{|G'(\phi + ih)|^2} + 2g \text{ Im } G(\phi + ih) = u^2 + 2ga , \qquad -\infty < \phi < \infty .$$

We can now use the transformation

$$w = m(\zeta)$$

to map S into any domain we think may be helpful for the analysis. (For a strictly numerical analysis, a mapping of S into a rectangle offers some advantages.) In a previous report, [1], the author used

$$w = m(\zeta) = \frac{h}{\pi} \ell n \left(- \frac{1}{\zeta}\right) .$$

This maps S into the upper half of the ζ-plane and gives

$$G(w) = G[\frac{h}{\pi} \ell n \ (-\frac{1}{\zeta})] = \Omega(\zeta) = ai + a\omega(\zeta) \ .$$

In [1] a nonlinear integral equation was derived for the determination of $\omega(\zeta)$ and then approximations were found for $G(w)$.

In the beginning the problem was to determine a function $F(z)$ analytic in an unknown domain D but subject to two boundary conditions along the unknown part of the boundary of D. We have seen how the dual boundary conditions enable us to transform the problem into one for the determination of a function $\Omega(\zeta)$ analytic in a known domain but subject to a nonlinear boundary condition. The nature of the nonlinearity is such that the probability of finding $\Omega(\zeta)$ in closed form is practically nil. We can only expect to find approximations for $\Omega(\zeta)$ and hence $G(w)$. Note however that such approximations present approximations for the equations of Γ_1 and Γ_2 in terms of the parameter ϕ. For example, if $\tilde{G}(w)$ is an approximation to $G(w)$ then the approximate parametric equations for Γ_2 are

$$\tilde{x} = \mathrm{Re}\ \tilde{G}(\phi + ih)$$

$$\tilde{y} = \mathrm{Im}\ \tilde{G}(\phi + ih) \ .$$

This parametric representation is not an advantage when it comes to calculating the depth of the liquid at some prescribed distance from the edge of the overfall. One such depth, the brink depth b in Figure 2.1, is perhaps the most important single numerical value which is desired from an analytis of the overfall. In order to calculate this depth we must at least approximate the solution of the transcendental equation

$$0 = \mathrm{Re}\, \widehat{G}(\phi + ih) .$$

This kind of burden is of course inherent in the above kind of analysis and it must be carried whenever we wish to estimate hydrodynamic quantities at prescribed points in the overfall.

Instead of using ϕ and ψ as independent variables for the flow we can introduce the dimensionless quantities $\xi = x/a$, $f = y/a$, $\psi(x,y)/au = \psi(a\xi, af)/au = \eta$; and then use ξ and η as independent variables with f as the basic dependent quantity. The equation

$$\psi(a\xi, af) = \eta au \tag{2.5}$$

defines the stream lines implicitly. However, since there is no stagnation point in the flow we can solve (2.5) for af. This gives

$$\frac{y}{a} = f(\xi, \eta) \tag{2.6}$$

which, since it presents the stream lines in explicit form, can be called the <u>stream line</u> function. The range of ξ is from minus infinity to plus infinity and the range of η is from zero to one.

The dimensionless velocity components

$$\frac{v_1}{u} = \frac{\psi_y(x,y)}{u} = a\eta_y$$

and

$$\frac{v_2}{u} = -\frac{\psi_x(x,y)}{u} = -a\eta_x$$

can be found in terms of f by differentiating (2.6) with respect to y and x. We have

$$\frac{1}{a} = f_n(\xi,\eta)\eta_y \tag{2.7}$$
$$0 = \frac{f_\xi(\xi,\eta)}{a} + f_\eta(\xi,\eta)\eta_x$$

which give the dimensionless horizontal component of velocity

$$\frac{v_1}{u} = a\eta_y = \frac{1}{f_\eta(\xi,\eta)} \tag{2.8}$$

and the dimensionless vertical component

$$\frac{v_2}{u} = -a\eta_x = \frac{f_\xi(\xi,\eta)}{f_\eta(\xi,\eta)} \; . \tag{2.9}$$

The equation for f can be found by recalling that

$$\eta_{xx} + \eta_{yy} = 0 \; .$$

It turns out that f must satisfy the nonlinear equation

$$\frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) - \frac{1}{2}\frac{\partial}{\partial\eta}\left(\frac{1+f_\xi^2}{f_\eta^2}\right) = 0 \tag{2.10}$$

or what is the same equation

$$(1+f_\xi^2)f_{\eta\eta} + f_\eta^2 f_{\xi\xi} - 2f_\xi f_\eta f_{\xi\eta} = 0 \; , \tag{2.11}$$

in the strip S^* shown in Figure 2.3. Along L_1^*, $f(\xi,\eta)$ must satisfy

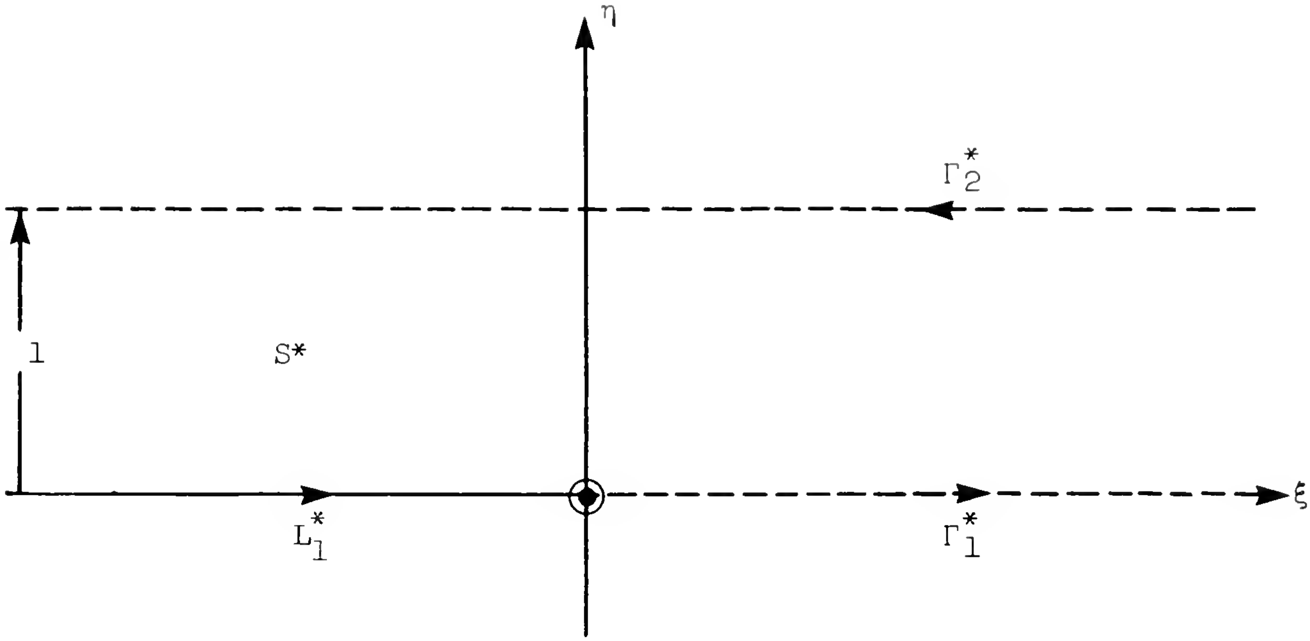

Fig. 2.3

$$f(\xi, 0) = 0 \quad (2.12)$$

Along Γ_1^* and Γ_2^*, $f(\xi,\eta)$ must satisfy the condition which comes from (2.2), namely,

$$1 + f_\xi^2 + (2\varepsilon f - 1 - 2\varepsilon) f_\eta^2 = 0 \quad (2.13)$$

where

$$\varepsilon = \frac{ga}{u^2}$$

is small when the flow is supercritical.

If ε is zero the equation (2.11) and the boundary conditions (2.12) and (2.13) are satisfied by

$$f(\xi,\eta) = \eta \; .$$

This corresponds to a simple parallel flow with velocity components

$$v_1 = u$$

$$v_2 = 0$$

and stream lines

$$y_o = a\eta$$

If ε is small a plausible assumption is that $f(\xi,\eta) \equiv f(\xi,\eta;\varepsilon)$ can be expanded in the form

$$\frac{y(\xi,\eta;\varepsilon)}{a} = f(\xi,\eta;\varepsilon) = \eta + \sum_{k=1}^{\infty} \varepsilon^k f_k(\xi,\eta) \tag{2.14}$$

which is at least asymptotically valid provided ξ is bounded away from plus infinity where f is infinite. Our immediate objective now is to analyze some of the consequences of the above assumption.

3. First Order Approximation

If we substitute (2.14) in (2.11) and the boundary conditions (2.12), (2.13) and then equate coefficients of like powers of ε we are led to a sequence of linear elliptic boundary value problems for the f_k's.

The function f_1 must satisfy the equation

$$f_{1\xi\xi}(\xi,\eta) + f_{1\eta\eta} = 0 \,, \qquad -\infty < \xi < \infty \,,\; 0 < \eta < 1 \tag{3.1}$$

and the boundary conditions

$$\begin{cases} f_1(\xi,0) = 0 \,, & \xi < 0 \,, \\ f_{1\eta}(\xi,0) = -1 \,, & 0 < \xi \,, \\ f_{1\eta}(\xi,1) = 0 \,, & -\infty < \xi < \infty \,. \end{cases} \tag{3.2}$$

The function f_2 must satisfy the equation

$$f_{2\xi\xi}(\xi,\eta) + f_{2\eta\eta} = 2(f_{1\xi}f_{1\xi\eta} + f_{1\eta}f_{1\eta\eta}) \,, \qquad -\infty < \xi < \infty \,,\; 0 < \eta < 1 \tag{3.3}$$

and the boundary conditions

$$\begin{cases} f_2(\xi,0) = 0 \,, & \xi < 0 \,, \\ f_{2\eta}(\xi,0) = f_1(\xi,0) + \frac{1}{2} f_{1\xi}^2(\xi,0) + \frac{3}{2} \,, & \xi > 0 \,, \\ f_{2\eta}(\xi,1) = f_1(\xi,1) + \frac{1}{2} f_{1\xi}^2(\xi,1) \,, & -\infty < \xi < \infty \,. \end{cases} \tag{3.4}$$

If we introduce the complex variable

$$\zeta = \xi + i\eta ,$$

then

$$f_1(\xi,\eta) = \text{Re } F_1(\zeta)$$

where $F_1(\zeta)$ is analytic when ζ is in S^*. Along the boundary of S^* $F_1(\zeta)$ must satisfy the conditions

$$\text{Re } F_1'(\xi) = 0 , \qquad \xi < 0 ,$$

$$\text{Im } F_1'(\xi) = 1 , \qquad \xi > 0 ,$$

$$\text{Im } F_1'(\xi+i) = 0 , \qquad -\infty < \xi < \infty .$$

Since the velocity at the brink of the overfall is neither zero nor infinite, $F_1'(\zeta)$ must be bounded at $\zeta = 0$. We suppose that the singular behavior of $F_1'(\zeta)$ as $\xi \to +\infty$ is as mild as possible. An analysis then shows that the function which satisfies the above conditions is

(3.5)
$$F_1(\zeta) = \frac{1}{\pi}\int_0^{\zeta} \ell n\left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1}\right) d\lambda .$$

The first order approximation to the stream line function is

(3.6)
$$\frac{y_1(\xi,\eta;\varepsilon)}{a} = \eta + \varepsilon f_1(\xi,\eta)$$

$$= \eta + \frac{\varepsilon}{\pi}\text{ Re}\int_0^{\zeta} \ell n\left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1}\right) d\lambda .$$

As far as we know the integral in (3.6) is neither tabulated nor expressible as a finite combination of tabulated functions. However, various pertinent expansions and estimations can be found without much trouble.

The representation (3.6) can be expressed in the form

$$\frac{y_1}{a} = \eta + \frac{\varepsilon}{\pi} \operatorname{Re} \int_0^{\zeta} [\ell n\,(-e^{-\pi\lambda}) - 2\,\ell n\,(\sqrt{1-e^{-\pi\lambda}} + 1)]\,d\lambda$$

$$= \eta + \frac{\varepsilon}{\pi} \operatorname{Re} \left\{ \begin{array}{l} \displaystyle\int_0^{\zeta} [-\pi(\lambda - i) - 2\,\ell n\, 2]\,d\lambda \\ \displaystyle - 2\int_0^{\infty} \ell n \left(\frac{\sqrt{1-e^{-\pi\lambda}} + 1}{2} \right) d\lambda \\ \displaystyle + 2\int_{\zeta}^{\infty} \ell n \left(\frac{\sqrt{1-e^{-\pi\lambda}} + 1}{2} \right) d\lambda \end{array} \right\} .$$

Since

$$-2\int_0^{\infty} \ell n \left(\frac{\sqrt{1-e^{-\pi\lambda}} + 1}{2} \right) d\lambda = -\frac{2}{\pi}\int_0^1 \ell n \left(\frac{\sqrt{1-t} + 1}{2} \right) \frac{dt}{t}$$

$$= -\frac{2}{\pi}\int_0^1 \frac{\ell n\,(1-\sigma^2)}{1+\sigma}\,d\sigma$$

$$= -\frac{2}{\pi}\,\ell n^2 2 + \frac{\pi}{6}$$

we have

$$\frac{y_1}{a} = \eta + \frac{\varepsilon}{\pi} \operatorname{Re} \left\{ \begin{array}{l} \displaystyle -\frac{\pi\zeta^2}{2} + (\pi i - 2\,\ell n\, 2)\zeta \\ \displaystyle -\frac{2}{\pi}\,\ell n^2 2 + \frac{\pi}{6} + 2\int_{\zeta}^{\infty} \ell n \left(\frac{\sqrt{1-e^{-\pi\lambda}} + 1}{2} \right) d\lambda \end{array} \right\} .$$

For Re $\zeta = \xi$ positive and large, the integral in the last expression

is small. Therefore, far downstream, we have the approximation

$$y_1 \sim a\eta + \frac{ga^2}{u^2\pi}\left\{\begin{array}{l} -\frac{\pi}{2}\xi^2 + \frac{\pi}{2}\eta^2 - (2\ \ell n\ 2)\xi \\ \\ -\pi\eta - \frac{2}{\pi}\ell n^2 2 + \frac{\pi}{6} \end{array}\right\}$$

which shows that the stream lines are parabolic. In terms of the original abscissa $x = \xi a$ the parabolas are given by

$$y_1 \sim a\eta + \frac{ga^2}{u^2\pi}\left\{\begin{array}{l} -\frac{\pi}{2}\frac{x^2}{a^2} + \frac{\pi}{2}\eta^2 - (2\ \ell n\ 2)\frac{x}{a} \\ \\ -\pi\eta - \frac{2}{\pi}\ell n^2 2 + \frac{\pi}{6} \end{array}\right\} .$$

If we let $a \rightarrow 0$ we find that the parabolas reduce to

$$y_1 = -\frac{gx^2}{2u^2}$$

which gives the shape of a thin heavy sheet of fluid projected horizontally from the brink with velocity u.

The first order approximation to the equation for the lower nappe of the overfall is

$$(3.7) \qquad \frac{y_1}{a} = \frac{\varepsilon}{\pi}\,\mathrm{Re}\int_0^{\xi} \ell n\left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1}\right)d\lambda\ , \qquad 0 < \xi\ .$$

If we use the substitution $\sqrt{1 - e^{-\pi\lambda}} = \sigma$, the representation (3.7) becomes

$$\frac{y_1}{a} = \frac{2\varepsilon}{\pi^2} \int_0^{\sqrt{1-e^{-\pi\xi}}} \frac{\sigma}{1-\sigma^2} \ell n \left(\frac{1-\sigma}{1+\sigma}\right) d\sigma$$

which shows that the equation of the lower nappe can be expanded in powers of $\sqrt{1 - e^{-\pi\xi}}$ if $0 \leq \xi < \infty$. We find that for the neighborhood of the brink the first order approximation for the lower nappe is

$$\frac{y_1}{a} = - \frac{4\varepsilon}{\pi^2} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{2k-1} \right) \frac{(1- e^{-\pi\xi})^{(2n+1)/2}}{2n+1} . \tag{3.8}$$

From this approximation, the slope of the lower nappe is zero at the brink but its curvature is infinite there.

The first order approximation for the brink depth ratio

$$\frac{b}{a} = f(0,1;\varepsilon)$$

is

$$\frac{y_1(0,1;\varepsilon)}{a} = 1 + \frac{\varepsilon}{\pi} \operatorname{Re} \int_0^{i} \ell n \left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} +1} \right) d\lambda . \tag{3.9}$$

This is the same as

$$\frac{y_1(0,1;\varepsilon)}{a} = 1 + \frac{\varepsilon}{\pi}\,\mathrm{Re}\left\{\begin{aligned} &\int_{-\infty+i}^{i} \ln\left(\frac{\sqrt{1-e^{-\pi\lambda}}-1}{\sqrt{1-e^{-\pi\lambda}}+1}\right)d\lambda \\ &-\int_{-\infty}^{0} \ln\left(\frac{\sqrt{1-e^{-\pi\lambda}}-1}{\sqrt{1-e^{-\pi\lambda}}+1}\right)d\lambda \end{aligned}\right\}$$

$$= 1 + \frac{\varepsilon}{\pi}\int_{-\infty}^{0} \ln\left(\frac{\sqrt{1+e^{-\pi\lambda}}-1}{\sqrt{1+e^{-\pi\lambda}}+1}\right)d\lambda$$

$$= 1 + \frac{\varepsilon}{\pi}\int_{-\infty}^{0} \frac{\lambda\pi\, d\lambda}{\sqrt{1+e^{-\pi\lambda}}}$$

$$= 1 - \frac{\varepsilon}{\pi^2}\int_{1}^{\infty} \frac{\ln\sigma\, d\sigma}{\sigma\sqrt{1+\sigma}}$$

$$= 1 - \frac{\varepsilon}{\pi^{5/2}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{n!}\int_{1}^{\infty}\frac{\ln\sigma\, d\sigma}{\sigma^{n+(3/2)}}$$

$$= 1 - \frac{\varepsilon}{\pi^{5/2}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{n!\,(n+\frac{1}{2})^2}$$

$$\frac{y_1(0,1;\varepsilon)}{a} = 1 - \varepsilon(.386)\ . \tag{3.10}$$

With this, the first order equation for the upper nappe is

$$\frac{y_1}{a} = 1 + \frac{\varepsilon}{\pi} \operatorname{Re} \int_0^{\xi+i} \ell n \left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1} \right) d\lambda$$

$$= 1 + \frac{\varepsilon}{\pi} \operatorname{Re} \left\{ \begin{array}{l} \displaystyle\int_0^{i} \ell n \left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1} \right) d\lambda \\ \displaystyle + \int_i^{\xi+i} \ell n \left(\frac{\sqrt{1 - e^{-\pi\lambda}} - 1}{\sqrt{1 - e^{-\pi\lambda}} + 1} \right) d\lambda \end{array} \right\}$$

$$\text{(3.11)} \qquad \frac{y_1}{a} = 1 - \varepsilon(.386) + \frac{\varepsilon}{\pi} \int_0^{\xi} \ell n \left(\frac{\sqrt{1 + e^{-\pi\sigma}} - 1}{\sqrt{1 + e^{-\pi\sigma}} + 1} \right) d\sigma \ .$$

The contraction coefficient b/a for the case $\varepsilon = 1$ is important because it appears in various formulas for flow measurement when the overfall is used as a metering device in a channel. We cannot expect the approximation (3.10) for the brink depth ratio to be good if ε is not small. However, it is interesting to see what it gives if the upstream velocity is critical. If $\varepsilon = 1$, (3.10) gives

$$\frac{y_1(0,1;1)}{a} = .614 \ .$$

This is to be compared with the value

$$\widetilde{\left(\frac{b}{a}\right)} = .708$$

which is the average of contraction coefficient values

.715	Rouse	[2]
.705	Southwell and Vaisey	[3]
.720	Jaeger	[4]
.720	Roy	[5]
.710	Fraser	[6]
.676	Hay and Markland	[7]

deduced from various experimental and mathematical approximation procedures different from the one we are using here. Each of the above papers covers the case of critical flow only. It should also be remarked that some of the values in the above table are not approximations to the actual brink depth ratio but are approximations to the stream depth ratio corresponding to $x = \varepsilon_b$ where ε_b is small. When this is the case the corresponding paper does not contain an estimate of just how small ε_b is.

The formula (3.10) for the brink depth ratio can be improved without actually finding the second order approximation f_2. One way to accomplish this is to replace $f(0,\eta;\varepsilon)$ with

(3.12) $$f^*(\eta;\varepsilon) = c_o + c_1(\eta-1) + \frac{c_2}{2}(\eta-1)^2$$

where the coefficients are chosen so that

$$f^*(0;\varepsilon) = f(0,0;\varepsilon)$$

$$f^*_\eta(0;\varepsilon) = f_\eta(0,0;\varepsilon)$$

which are known values; and so that

(3.13) $$f_\eta^*(1;\varepsilon) = \sqrt{\frac{1+\varepsilon^2 f_{1\xi}^2(0,1)}{1-2\varepsilon^2 f_1(0,1)}} \quad .$$

The right member of (3.13) comes from the boundary condition (2.13), namely,

(3.14) $$f_\eta^2(0,1;\varepsilon) = \frac{1+f_\xi^2(0,1;\varepsilon)}{1+2\varepsilon-2\varepsilon f(0,1;\varepsilon)}$$

after $f(\xi,\eta;\varepsilon)$ in the right-hand side of (3.14) is replaced by

$$f(\xi,\eta;\varepsilon) = \eta + \varepsilon f_1(\xi,\eta) \ .$$

Using

$$f_{1\xi}(0,1) = \frac{1}{\pi} \operatorname{Re} \ \ell n \left(\frac{\sqrt{1-e^{-\pi\zeta}}-1}{\sqrt{1-e^{-\pi\zeta}}+1} \right) \Bigg|_{\zeta=i}$$

$$= \frac{1}{\pi} \ \ell n \ \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

and

$$f_1(0,1) = -.386$$

a computation of the right side of (3.13) shows

$$f_\eta^*(1;\varepsilon) = \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}} \ .$$

The value $f_\eta(0,0;\varepsilon)$ also comes from (2.13) and it is

$$f_\eta(0,0;\varepsilon) = \frac{1}{\sqrt{1+2\varepsilon}} \ .$$

The coefficients in (3.12) are then determined by

$$f^*(0;\varepsilon) = c_o - c_1 + \frac{c_2}{2} = 0 ,$$

$$f^*_\eta(0;\varepsilon) = \qquad c_1 - c_2 = \frac{1}{\sqrt{1+2\varepsilon}} ,$$

$$f^*_\eta(1;\varepsilon) = \qquad c_1 \qquad = \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}}$$

from which the improved approximation to the brink depth ratio is

$$f^*(1;\varepsilon) = c_o = c_1 = \frac{c_2}{2} ,$$

$$= c_1 - \frac{1}{2}\left[c_1 - \frac{1}{\sqrt{1+2\varepsilon}}\right] ,$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{1+2\varepsilon}} + c_1\right] ,$$

$$f^*(1;\varepsilon) = \frac{1}{2}\left[\frac{1}{\sqrt{1+2\varepsilon}} + \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}}\right] . \tag{3.15}$$

This result can be expected to be a better approximation, for a wider range of values of ε, than the formula (3.10), namely,

$$\frac{y_1(0,1;\varepsilon)}{a} = 1 - \varepsilon(.386) .$$

For $\varepsilon = .1$ formula (3.10) gives

$$\frac{y_1(0,1;.1)}{a} = .961$$

while (3.15) gives

$$f^*(1;.1) = .955 .$$

For $\varepsilon = .5$ we have

$$\frac{y_1(0,1;.5)}{a} = .807$$

$$f^*(1;.5) = .829 \ .$$

For $\varepsilon = .9$ which means that the flow is supercritical to the extent that $u = \frac{\sqrt{10}}{3}\sqrt{ga} = 1.05\sqrt{ga}$ we find

$$\frac{y_1(0,1;.9)}{a} = .653$$

$$f^*(1;.9) = .738 \ .$$

For $\varepsilon = 1$, corresponding to critical speed we have

$$\frac{y_1(0,1;1)}{a} = .614$$

while

$$f^*(1;1) = .719 \ .$$

This last result compares very favorably with the average

$$\left(\frac{\tilde{b}}{a}\right) = .708$$

of brink depth ratios from other sources cited above.

Better approximations can presumably be found by passing to a determination of f_2 but for the time being the results presented above appear to be sufficiently accurate.

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13. ABSTRACT

This report contains a first order analysis of the supercritical flow in an overfall based on the use of the stream line function as the basic dependent variable. It presents approximate equations for the nappes; and the formula

$$f^*(1;\varepsilon) = \frac{1}{2}\left[\frac{1}{\sqrt{1+2\varepsilon}} + \sqrt{\frac{1+\varepsilon^2(.314)}{1+\varepsilon^2(.772)}}\right]$$

where

$$\varepsilon = \frac{ga}{u^2},$$

g is the gravitational acceleration, a is the upstream depth and u is the velocity there. This formula is a good approximation to the brink depth ratio b/a for the range $0 \leq \varepsilon \leq 1$.

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